Stochastic Processes and their Applications 32 (1989) 281-287
North-Holland

MOMENT GENERATING FUNCTION OF A FIRST HITTING PLACE FOR THE INTEGRATED ORNSTEIN-UHLENBECK PROCESS

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Received 18 April 1988
Revised 18 October 1988

We consider the two-dimensional process $(x(t), y(t))$, where $y(t) = \mathrm{d}x(t)/\mathrm{d}t$ is an Ornstein-Uhlenbeck process. Let T be the smallest t for which $y(t) = v$. In this note we obtain an explicit expression, in terms of a parabolic cylinder function, for the moment generating function or the characteristic function of $x(T)$ and we evaluate the expected value of $x(T)$.

first hitting place * Ornstein-Uhlenbeck process * parabolic cylinder function

1. Introduction

First passage distributions of stochastic processes are needed in many applications, especially in the physical and biological sciences. However, few explicit results have been obtained so far for processes of dimension greater than one: the two-dimensional Brownian motion, for example, has been studied, in particular, by Buckholtz and Wasan (1979) who derived a formula for a first passage time density, and more extensively by Iyengar (1985). Wendel (1980) considered the n-dimensional Brownian motion inside, outside and between spheres in $\mathbb{R}^n$. Lefebvre (1987) has obtained a first passage density for the integrated Brownian motion, a process which had been studied earlier by McKean Jr. (1963), Goldman (1971) and Gor'kov (1975).

Other explicit results have surely been derived, but, in general, evaluating statistics of first hitting times and places for higher-order processes is a difficult problem. Here, we consider the two-dimensional process

$$\begin{aligned} \mathrm{d}x(t) &= y(t)\,\mathrm{d}t, \\ \mathrm{d}y(t) &= -y(t)\,\mathrm{d}t + \mathrm{d}W(t), \end{aligned} \tag{1.1}$$

Research supported by the Natural Sciences and Engineering Research Council of Canada, grant no. OGP0007989.

where $W(t)$ is a Wiener process with zero drift and, for simplicity, variance parameter equal to 2. Thus, $y(t)$ is an Ornstein-Uhlenbeck process. Let

$$T=\min\{t;\ y(t)=v\}. \tag{1.2}$$

The moment generating function (m.g.f.) of T is well-known; see Prabhu (1965, p. 107), for instance. In this note, we are interested in the value of $x(T)$; that is, the value of $x(t)$ when the process $y(t)$ hits the line $y=v$ for the first time. In Section 2, we obtain an explicit expression, in terms of a parabolic cylinder function, for the m.g.f. of $x(T)$ when $y(0)\geqslant v\geqslant 0$, and then for its characteristic function in the case $y(0)\leqslant v$ (with $v>0$). In Section 3, we evaluate the expected value of $x(T)$, which turns out to be the same in both cases considered in Section 2.

2. Moment generating function of $x(T)$

Let $(x(0), y(0))=(x, y)$ and suppose first that $y\geqslant v\geqslant 0$. Write

$$p(x, y; u, t)\,\mathrm{d}u\,\mathrm{d}t=P_{(x,y)}[x(T)\in \mathrm{d}u,\ T\in \mathrm{d}t], \tag{2.1}$$

where T $(=T(v))$ is defined in (1.2). Then, the function p satisfies the Kolmogorov backward equation

$$p_{yy}+yp_x-yp_y=p_t \tag{2.2}$$

for (x, y, t) in

$$C_0=\{(x, y, t):\ x<u,\ y>v,\ t>0\}. \tag{2.3}$$

Indeed, we deduce from (1.1) that when $y(0)\geqslant v\geqslant 0$ we must have $x(T)\geqslant x(0)$.

Next, let

$$f(x, y; u)\,\mathrm{d}u=P_{(x,y)}[x(T)\in \mathrm{d}u]. \tag{2.4}$$

Then, f satisfies the partial differential equation

$$f_{yy}+yf_x-yf_y=0 \tag{2.5}$$

for (x, y) in

$$C_1=\{(x, y):\ x<u,\ y>v\}. \tag{2.6}$$

Finally, write

$$g(a, y; u)=\int_{-\infty}^{u} \mathrm{e}^{ax} f(x, y; u)\,\mathrm{d}x, \tag{2.7}$$

where $a>0$. We find that

$$g_{yy}-ayg-yg_y=0 \tag{2.8}$$

for $y>v$. The boundary condition is

$$g(a, v; u)=\mathrm{e}^{au}. \tag{2.9}$$

To solve the ordinary differential equation (2.8), let

$$g(a, y; u) = h(z)\,\mathrm{e}^{-ay}, \tag{2.10}$$

where

$$z = y + 2a. \tag{2.11}$$

The function h satisfies

$$h_{zz} - zh_z + a^2 h = 0. \tag{2.12}$$

Next, put

$$h(z) = w(z)\exp(\tfrac{1}{4}z^2). \tag{2.13}$$

Then, we find that

$$w_{zz} = w[\tfrac{1}{4}z^2 - a^2 - \tfrac{1}{2}]. \tag{2.14}$$

The general solution of equation (2.14) is, setting $c = -a^2 - \frac{1}{2}$,

$$w(z) = k_1 U(c, z) + k_2 V(c, z), \tag{2.15}$$

where U and V are parabolic cylinder functions defined by (see Abramowitz and Stegun, 1965, p. 687)

$$U(c, z) = \cos[\pi(\tfrac{1}{4} + \tfrac{1}{2}c)]\,Y_1 - \sin[\pi(\tfrac{1}{4} + \tfrac{1}{2}c)]\,Y_2,$$

and

$$V(c, z) = \{\sin[\pi(\tfrac{1}{4} + \tfrac{1}{2}c)]\,Y_1 + \cos[\pi(\tfrac{1}{4} + \tfrac{1}{2}c)]\,Y_2\}/\Gamma(\tfrac{1}{2} - c),$$

with

$$Y_1 = \pi^{-1/2}\Gamma(\tfrac{1}{4} - \tfrac{1}{2}c)2^{-c/2-1/4}\exp(\tfrac{1}{4}z^2)$$
$$\times\{1 + (c - \tfrac{1}{2})z^2/2! + (c - \tfrac{1}{2})(c - \tfrac{5}{2})z^4/4! + \cdots\},$$

and

$$Y_2 = \pi^{-1/2}\Gamma(\tfrac{3}{4} - \tfrac{1}{2}c)2^{-c/2+1/4}\exp(\tfrac{1}{4}z^2)$$
$$\times\{z + (c - \tfrac{3}{2})z^3/3! + (c - \tfrac{3}{2})(c - \tfrac{7}{2})z^5/5! + \cdots\}.$$

Now, since

$$w(2a + y) = g(a, y; u)\exp[-\tfrac{1}{4}y^2 - a^2],$$

we deduce that $w(z)$ decreases to zero as y goes to $+\infty$. Using the asymptotic expansions of $U(c, z)$ and $V(c, z)$ for large z, we find that, as z goes to infinity, $U(c, z)$ tends to zero, but $V(c, z)$ diverges. Hence, we must set k_2 equal to zero in equation (2.15). Finally, using the boundary condition (2.9), we obtain the formula

$$g(a, y; u) = \mathrm{e}^{au}\exp[\tfrac{1}{4}(y^2 - v^2)]\,U(c, y + 2a)/U(c, v + 2a). \tag{2.16}$$

Lemma 1. *We have*

$$f(x, y; u) = f(0, y; u - x). \tag{2.17}$$

Proof. This follows at once from

$$x(T) = x(0) + \int_0^T y(t)\,\mathrm{d}t. \qquad \square \tag{2.18}$$

Using this lemma we can prove the theorem that follows.

Theorem 1. *When* $y(0) = y \geqslant v \geqslant 0$, *the moment generating function of* $x(T)$ *is given by*

$$M(x, y; b) = E_{(x,y)}[\mathrm{e}^{-bx(T)}] = g(b, y; -x), \tag{2.19}$$

where $b \geqslant 0$.

Proof. We have

$$M(x, y; b) = \int_x^\infty \mathrm{e}^{-bu} f(x, y; u)\,\mathrm{d}u.$$

Therefore, using Lemma 1 we may write, with $r = u - x$,

$$M(x, y; b) = \mathrm{e}^{-bx} \int_0^\infty \mathrm{e}^{-br} f(0, y; r)\,\mathrm{d}r. \tag{2.20}$$

But, we have

$$g(a, y; u) = \int_{-\infty}^u \mathrm{e}^{ax} f(x, y; u)\,\mathrm{d}x = \mathrm{e}^{au} \int_0^\infty \mathrm{e}^{-ar} f(0, y; r)\,\mathrm{d}r.$$

Formula (2.19) then follows from (2.20). $\square$

Next, we consider the case when $y(0) \leqslant v$, where v is now strictly positive. Let

$$q(x, y; u, t)\,\mathrm{d}u\,\mathrm{d}t = P_{(x,y)}[x(T) \in \mathrm{d}u, T \in \mathrm{d}t]. \tag{2.21}$$

Then the function q satisfies the Kolmogorov backward equation (2.2), but for (x, y, t) in

$$C_2 = \{(x, y, t): -\infty < x < \infty, y < v, t > 0\}, \tag{2.22}$$

since, in this case, $x(T)$ can take any real value (see (1.1)). Therefore, we may write

$$F_{yy} + yF_x - yF_y = 0 \tag{2.23}$$

for (x, y) in

$$C_3 = \{(x, y): -\infty < x < \infty, y < v\}, \tag{2.24}$$

where

$$F(x, y; u)\,\mathrm{d}u = P_{(x,y)}[x(T) \in \mathrm{d}u]. \tag{2.25}$$

Next, since x can be any real number, we define

$$G(\alpha, y; u) = \int_{-\infty}^{\infty} e^{i\alpha x} F(x, y; u)\, dx, \tag{2.26}$$

where $-\infty < \alpha < \infty$. From (2.23), we deduce that G satisfies the ordinary differential equation

$$G_{yy} - yG_y - i\alpha y G = 0, \tag{2.27}$$

for $y < v$, subject to the boundary condition

$$G(\alpha, v; u) = e^{i\alpha u}. \tag{2.28}$$

If we make the substitutions

$$G(\alpha, y; u) = H(z)\, e^{-i\alpha y}, \tag{2.29}$$

where

$$z = y + 2i\alpha, \tag{2.30}$$

and

$$H(z) = N(z) \exp(\tfrac{1}{4}z^2), \tag{2.31}$$

we obtain

$$N_{zz} = N[\tfrac{1}{4}z^2 + \alpha^2 - \tfrac{1}{2}]. \tag{2.32}$$

Two linearly independent solutions of equation (2.32) are the parabolic cylinder functions $U(\alpha^2 - \frac{1}{2}, z)$ and $V(\alpha^2 - \frac{1}{2}, z)$. With $\alpha = 0$, we obtain

$$N(y) = \exp(-\tfrac{1}{4}y^2). \tag{2.33}$$

Indeed, since Lemma 1 is still valid when $y \leqslant v$ we may write

$$\begin{aligned} N(y) &= \exp(-\tfrac{1}{4}y^2) \int_{-\infty}^{\infty} F(x, y; u)\, dx \\ &= \exp(-\tfrac{1}{4}y^2) \int_{-\infty}^{\infty} P_{(0,y)}[x(T) \in dr] \\ &= \exp(-\tfrac{1}{4}y^2), \end{aligned}$$

the last equality holding since $P[T < \infty] = 1$ (see Cox and Miller, 1965, p. 234).

Now, we have

$$U(-\tfrac{1}{2}, y) = \exp(-\tfrac{1}{4}y^2) \tag{2.34}$$

and

$$V(-\tfrac{1}{2}, y) = (2/\pi)^{1/2} \exp(-\tfrac{1}{4}y^2)\{y + y^3/3! + 3y^5/5! + (3)(5)y^7/7! + \cdots\}. \tag{2.35}$$

Hence, we conclude that we must again eliminate the solution $V(\alpha^2-\frac{1}{2}, z)$. Thus, from the boundary condition (2.28) we deduce that

$$G(\alpha, y; u)=\mathrm{e}^{\mathrm{i}\alpha u}\exp[\tfrac{1}{4}(y^2-v^2)]U(\alpha^2-\tfrac{1}{2}, y+2\mathrm{i}\alpha)/U(\alpha^2-\tfrac{1}{2}, v+2\mathrm{i}\alpha). \quad (2.36)$$

Finally, using once again Lemma 1 we obtain the following theorem.

Theorem 2. *When $y(0)=y\leqslant v$ (>0), the characteristic function of $x(T)$ is given by*

$$E_{(x,y)}[\mathrm{e}^{-\mathrm{i}\beta x(T)}]=\mathrm{e}^{-\mathrm{i}\beta x}\exp[\tfrac{1}{4}(y^2-v^2)]U(\beta^2-\tfrac{1}{2}, y+2\mathrm{i}\beta)/U(\beta^2-\tfrac{1}{2}, v+2\mathrm{i}\beta), \quad (2.37)$$

where β is a real parameter. □

3. Expected value of $x(T)$

To conclude, in this section we shall make use of the m.g.f. of $x(T)$ given in Theorem 1 to evaluate the expected value of $x(T)$. We have the result that follows.

Theorem 3. *When $y(0)=y\geqslant v\geqslant 0$, the expected value of $x(T)$ is*

$$E[x(T)]=x+y-v. \quad (3.1)$$

Proof. Theorem 1 tells us that

$$E_{(x,y)}[\mathrm{e}^{-bx(T)}]=\mathrm{e}^{-bx}\exp[\tfrac{1}{4}(y^2-v^2)]U(-b^2-\tfrac{1}{2}, y+2b)/U(-b^2-\tfrac{1}{2}, v+2b),$$

where

$$\begin{aligned}U(-b^2-\tfrac{1}{2}, y+2b)&=\pi^{-1/2}2^{b^2/2}\exp[-\tfrac{1}{4}(y+2b)^2]\\&\quad\times\{\cos(\tfrac{1}{2}\pi b^2)\Gamma[\tfrac{1}{2}(b^2+1)]M[-\tfrac{1}{2}b^2, \tfrac{1}{2}, \tfrac{1}{2}(y+2b)^2]\\&\quad+2^{1/2}\sin(\tfrac{1}{2}\pi b^2)\Gamma[\tfrac{1}{2}(b^2+2)](y+2b)M[\tfrac{1}{2}(1-b^2), \tfrac{3}{2}, \tfrac{1}{2}(y+2b)^2]\},\end{aligned}$$

the function $M[x, y, z]$ being a confluent hypergeometric function defined in Abramowitz and Stegun (1965, p. 504) (see also above). Differentiating and letting b decrease to zero we find, after some computation, that

$$\lim_{\substack{b\to 0\\ b>0}} \mathrm{d}U(-b^2-\tfrac{1}{2}, y+2b)/\mathrm{d}b=-y\exp(-\tfrac{1}{4}y^2). \quad (3.2)$$

Since we know by (2.34) that

$$U(-\tfrac{1}{2}, y)=\exp(-\tfrac{1}{4}y^2),$$

we easily obtain formula (3.1). □

From Theorem 2 we deduce this corollary.

Corollary 1. *In the case $y(0)=y\leqslant v$ (>0), the expected value of $x(T)$ is also given by*

$$E[x(T)]=x+y-v. \qquad \square \quad (3.3)$$

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